<u>Remark</u> 1) One can also define Hecke operators as operators on functions of lattices

Let $\mathcal{L}$ = set of all lattices in $\mathbb{C}$

$$\{ \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \mid \omega_1, \omega_2 \in \mathbb{C} \}$$
ω_1, ω_2 lin indep / $\mathbb{R}$

and $\mathbb{Z}[\mathcal{L}]$ = free abelian gp. generated by all lattices

Recall any homog. of degree $-k$ function F on lattices $F : \mathcal{L} \to \mathbb{C}$ gives rise to a modular form f of wt k via

$$f(\tau) := F(\mathbb{Z}\tau + \mathbb{Z})$$

<u>Defn</u> The n-th Hecke operator T_n on lattices is defined as follows

$$T_n : \mathbb{Z}[\mathcal{L}] \longrightarrow \mathbb{Z}[\mathcal{L}], \text{ For } \Lambda \in \mathcal{L}$$

~~define~~ $$T_n \Lambda := \sum_{[\Lambda : \Lambda'] = n} \Lambda'$$

and extend it linearly on $\mathbb{Z}[\mathcal{L}]$.

Note the sum is finite since any such Λ' satisfy $n\Lambda \subseteq \Lambda' \subseteq \Lambda$, so corresponds to a s/gp of order n of $\Lambda / n\Lambda \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$

To see $n\Lambda \subseteq \Lambda'$ note that every elt of Λ/Λ' has order dividing n.

If $l \in \Lambda$ then $n(l + \Lambda') = e_{\Lambda/\Lambda'} = \Lambda'$

Hence $nl \in \Lambda'$ and $n\Lambda \subseteq \Lambda'$

For $F : \mathcal{L} \longrightarrow \mathbb{C}$ homog of degree $-k$

define $$T_n F(\Lambda) := n^{k-1} \sum_{[\Lambda : \Lambda'] = n} F(\Lambda')$$

Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\mathrm{Im}(\omega_2/\omega_1) > 0$

and $\Lambda' = \mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2'$ " $\mathrm{Im}(\omega_2'/\omega_1') > 0$

be a sublattice of index n.

If we write $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$ then

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = n.$

General properly ordered basis of Λ' is obtained from $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$ by left multiplication by a member of $SL_2(\mathbb{Z}) = \Gamma$

Thus the right cosets $\Gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ describes all matrices leading from a basis $\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$ of Λ to a basis $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$ of Λ'.

Let $M(n)$ = integral matrices of det n as before

We have $M(n) = \sum_{\sigma_1 \in R} SL_2(\mathbb{Z})\,\sigma_1$

R = set of reps for right cosets of $SL_2(\mathbb{Z})$ in $M(n)$.

<u>Lemma</u> Let $R := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, \begin{array}{l} ad = n \quad a \geq 1 \\ 0 \leq b < d \end{array} \right\}$

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

For $\sigma \in R$, let $\Lambda_\sigma := \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ where

$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \sigma\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$. Then $R \to \Lambda(n) = \{\Lambda' \subset \Lambda \mid [\Lambda : \Lambda'] = n\}$

$\sigma \to \Lambda_\sigma$

is a bijection from R onto the set $\Lambda(n)$ of sublattices of index n in Λ.

Now let $\Lambda = \Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$

Let $\Lambda' \subset \Lambda$ be a sublattice of index n corresponding to $\alpha_i \in SL_2(\mathbb{Z}) \backslash M(n)$

where $M(n) = \bigcup_{i=1}^{\ell} \Gamma \alpha_i$. Let $\alpha_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

ie $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \alpha_i \begin{pmatrix} \tau \\ 1 \end{pmatrix}$ is a basis of Λ'

$$F(\Lambda') = F\left(\omega_1'\left(\mathbb{Z} + \mathbb{Z}\frac{\omega_2'}{\omega_1'}\right)\right)$$

$$= \omega_1'^{-k} f\left(\omega_2'/\omega_1'\right) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

$$= n^{-k/2}\left(f\Big|_k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau)$$

$$(T_n F)(\Lambda_\tau) = n^{k-1} \sum_{[\Lambda_\tau : \Lambda'] = n} F(\Lambda']$$

$$= n^{k-1} n^{-k/2} \sum_{i=1}^{\ell} f_k | \alpha_i$$

$$= n^{k/2 - 1} \sum_{\alpha_i \in \Gamma \backslash M(n)} f | \alpha_i = (T_n f)(\tau)$$

as before.

Rmk ② There is a 3rd way to look at Hecke operators ; in terms of double cosets.

Let H_1, H_2 be 2 groups acting on a set X on the left and on the right resp. so that the actions are compatible ie

$$(h_1 x) \cdot h_2 = h_1 (x h_2) \quad \forall h_1 \in H_1, \forall h_2 \in H_2, \forall x \in X$$

The set of orbits is denoted by $H_1 \backslash X / H_2$

If H_1, H_2 are 2 s/gps of a group G

$H_1 \backslash G / H_2$ is the set of double cosets $H_1 g H_2$

One may think this as H_1 acting on G/H_2 by left translation. And $H_1 \backslash G/H_2$ is simply the set of orbits of this action.

One can prove (Bump's book)

Prop: Let $\alpha \in GL(2, \mathbb{Q})^+$. Then the double coset $\Gamma \alpha \Gamma$ is a finite union of right cosets.

ie $\quad \Gamma \alpha \Gamma = \bigcup_{i=1}^{N} \Gamma \alpha_i \qquad \alpha_i \in GL_1(2, \mathbb{Q})^+$

and $N = [\Gamma : (\alpha^{-1} \Gamma \alpha) \cap \Gamma]$

$\left(GL_2(\mathbb{Q})^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Q}) \mid \det > 0 \right\} \right)$

(6.22)

<u>Defn</u> A Hecke operator T_α acting on $M_k(\Gamma)$ is defined as

$$T_\alpha f = \sum f|\alpha_i \quad \text{for each } \alpha \in GL(2,\mathbb{Q})$$

where $\Gamma\alpha\Gamma = \bigcup_{i=1}^{N} \Gamma\alpha_i$

This gives an action of $\Gamma\alpha\Gamma$ on $M_k(\Gamma)$

$\mathbb{T}$ = free abelian group generated by symbols T_α as α runs over a complete set of reps for $\Gamma\backslash GL_2(\mathbb{Q})^+/\Gamma$.

Define a multiplication on $\mathbb{T}$ by

$$T_\alpha \circ T_\beta = \sum_{\sigma \in \Gamma\backslash GL_2(\mathbb{Q})^+/\Gamma} m(\alpha,\beta:\sigma)\, T_\sigma$$

where $m(\alpha,\beta:\sigma) = \#\{(i,j) \mid \sigma \in \Gamma\alpha_i\beta_j\}$

where $\Gamma\alpha\Gamma = \bigcup \Gamma\alpha_i \qquad \Gamma\beta\Gamma = \bigcup \Gamma\beta_j$

Then $\circ$ is assoc., and commutative

$\mathbb{T}$ is called the Hecke algebra for $SL(2,\mathbb{Z})$

As a consequence of elementary division thm one has

<u>Prop:</u> A complete set of reps for $\Gamma \backslash GL_2(\mathbb{Q})^+ / \Gamma$ consists of diagonal matrices $\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$ with $d_1, d_2 \in \mathbb{Q}$, $d_1/d_2 \in \mathbb{Z}^+$

<u>Defn</u> $$T_n = \sum_{\substack{d_1 d_2 = n \\ d_2 \mid d_1 \\ d_1, d_2 \in \mathbb{Z}}} T_{\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}}$$

and $$T_n f := \sum T_{\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}} f$$

Since $\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$ with $d_1, d_2 \in \mathbb{Q}$, $d_1/d_2 \in \mathbb{Z}^+$ is a complete set of reps for $\Gamma \backslash GL_2(\mathbb{Q})^+ / \Gamma$

$$\Gamma \bigcup_{\substack{d_1 d_2 = n \\ d_2 \mid d_1 \\ d_i \in \mathbb{Z}}} \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \Gamma = M(n) = \bigcup_{\alpha_i \in \Gamma \backslash M(n)} \Gamma \alpha_i$$

and this leads to the usual defn from before up to factor $n^{k/2-1}$

Our next goal is to show that the Hecke operators, T_n's, are Hermitian wrt Petersson inner product. This will allow us to use linear algebra to conclude that $S_k(\Gamma)$ has a basis consisting of Hecke eigenforms.

More precisely <u>Recall</u>:

Let S be a finite dim'l inner product space over $\mathbb{C}$. Any linear operator $T: S \to S$ has an <u>adjoint</u> operator $T^*: S \to S$ defined by

$$\langle Tf, g \rangle = \langle f, T^* g \rangle$$

T is called <u>normal</u> if it commutes with T^*

T is called <u>self-adjoint</u> if $T = T^*$ or <u>Hermitian</u>

<u>Spectral thm</u>

<u>Thm of Lin Alg</u>: Let S be a finite dim'l inner product space over $\mathbb{C}$. Let Π be a commuting family of normal operators $T: S \to S$. Then $\exists$ an orthonormal basis $\mathcal{B}$ of S which consists of common eigenfunctions of all $T \in \Pi$.

Hence to use this thm we need to show that T_n's are normal wrt the Petersson inner product.
We'll show that T_n's are Hermitian.
This can be done more directly.
We'll do this by looking at the action of T_n on Poincaré series P_m. Since they generate cusp forms this will allow us to deduce the result for S_k

Thm 6.12 Let $k>2$ and

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m \gamma z) j(\gamma, z)^{-k}$$

the m-th Poincaré series. Then

$$T_n P_m = n^{k-1} \sum_{d \mid (n,m)} d^{1-k} P_{\frac{mn}{d^2}}$$

Proof

$$T_n P_m = n^{k/2-1} \sum_{\sigma \in \Gamma \backslash M(n)} P_m | \sigma$$

$$= n^{k/2-1} \sum_{\sigma} j(\sigma, z)^{-k} n^{k/2} P_m(\sigma z)$$

$$= n^{k-1} \sum_{\sigma \in \Gamma \backslash M(n)} j(\sigma, z)^{-k} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m \gamma \sigma z) j(\gamma, \sigma z)^{-k}$$

$$= n^{k-1} \sum_{\sigma} \sum_{\gamma} e(m\gamma\sigma z)\, j(\gamma\sigma, z)^{-k}$$

$$= n^{k-1} \sum_{S \in \Gamma_\infty \backslash M(n)} e(mSz)\, j(S, z)^{-k}$$

since if $\{\sigma_j\}_j$ is a set of reps for $\Gamma \backslash M(n)$

and $\{\gamma_i\}_i$ a set of reps for $\Gamma_\infty \backslash \Gamma$

then $\{\gamma_i \sigma_j\}$ is a set of reps for $\Gamma_\infty \backslash M(n)$.

$$\Gamma = \bigcup_i \Gamma_\infty \gamma_i \qquad M(n) = \bigcup_j \Gamma \sigma_j$$

$$M(n) = \bigcup_j \bigcup_i \Gamma_\infty \gamma_i \sigma_j$$

Now note that $\sigma_j \gamma_i$ is also a set of reps for $\Gamma_\infty \backslash M(n)$. To see this

let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(n) \qquad AD - BC = n$

We want to write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underbrace{\pm\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}}_{\in \Gamma_\infty} \underbrace{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}_{\in \Gamma_\infty \backslash M(n)} \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in \Gamma_\infty \backslash \Gamma}.$$

6.27

To do this we first find δ, γ s.t

$(\gamma, \delta) = 1$ and $C\delta - D\gamma = 0$.

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_p \backslash \Gamma$ be the matrix with second row (γ, δ)

Multiplying both sides by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}$

We can assume $C = 0$. and we have

$$\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b+dq \\ 0 & d \end{pmatrix}$$

We can now take $A' = a$, $D' = d$
choose q so that $B' - qD' = b$ is in our residue system mod d.

Hence we can write

$$T_n P_m = n^{k-1} \sum_{\sigma, \gamma} e(m\,\gamma\sigma z)\, j(\gamma\sigma, z)^{-k}$$

$$= n^{k-1} \sum_{\sigma, \gamma} e(m\,\sigma\gamma z)\, j(\sigma\gamma, z)^{-k}$$

$$= n^{k-1} \sum_{\sigma, \gamma} e(m\,\sigma(\gamma z))\, \underbrace{j(\sigma, \gamma z)^{-k}}_{d^{-k}\ \left(\text{since } \sigma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right)}\, j(\sigma, z)^{-k}$$

$$= n^{k-1} \sum_{\substack{ad = n \\ d > 0}} \sum_{b \bmod d} \sum_{\gamma} e\left(m\left(\frac{a\gamma z + b}{d}\right)\right) d^{-k}\, j(\gamma, z)^{-k}$$

$$= n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{\gamma} e\left(\frac{ma}{d} \gamma z\right) j(\gamma, z)^{-k} \underbrace{\sum_{b \bmod d} e\left(\frac{mb}{d}\right)}_{= \begin{cases} d & \text{if } d \mid m \\ 0 & \text{otherwise} \end{cases}}$$

$$= n^{k-1} \sum_{d \mid (n,m)} d^{1-k} \underbrace{\sum_{\gamma} e\left(\frac{mn}{d^2} \gamma z\right) j(\gamma, z)^{-k}}_{P_{mn/d^2}(z)}$$

Hence $T_n P_m = n^{k-1} \sum_{d \mid (n,m)} d^{1-k} P_{mn/d^2}$ ☒

<u>Cor 6.13</u> $T_n P_0 = n^{k-1} \sum_{d \mid (n,0)} d^{1-k} P_0$

$$= \sum_{d \mid n} \left(\frac{n}{d}\right)^{k-1} P_0 = \sigma_{k-1}(n) P_0$$

Hence $P_0 = E_k$ is an eigenfunction of T_n with e.value $\sigma_{k-1}(n)$ $\forall n$.

<u>Cor 6.14</u> $m^{k-1} T_n P_m = n^{k-1} T_m P_n$.

This follows from the sym in n, m in the Thm.

A similar symmetry holds for the inner product of $T_n f$ against P_m

<u>Prop 6.15</u> For $m, n \geq 1$, $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma)$

$$m^{k-1} \langle T_n f, P_m \rangle = n^{k-1} \langle T_m f, P_n \rangle$$

<u>Proof</u> Recall that if $g = \sum b_m q^m \in M_k$ then $\Gamma(k-1)^{-1} (4\pi m)^{k-1} \langle g, P_m \rangle = b_m$

Moreover if $g = T_n f$ then $b_m = \sum_{d | (m,n)} d^{k-1} a_{mn/d^2}$

and if $h = T_m f$ and $h = \sum c_n q^n$ then

$$c_n = \sum_{d | (m,n)} d^{k-1} a_{mn/d^2}$$

Hence $c_n = b_m$

Now $\Gamma(k-1)^{-1} (4\pi m)^{k-1} \langle T_n f, P_m \rangle = b_m$
$= c_n = \langle T_m f, P_n \rangle \cdot \Gamma(k-1)^{-1} (4\pi n)^{k-1}$

$$\Rightarrow m^{k-1} \langle T_n f, P_m \rangle = n^{k-1} \langle T_m f, P_n \rangle$$

$\blacksquare$

Combining these we will prove

<u>Thm 6.16</u> The Hecke operators acting on the space of cusp forms are self-adjoint wrt $\langle , \rangle$ - ie $\langle T_n f, g\rangle = \langle f, T_n g\rangle \quad \forall\, f, g \in S_k(\Gamma)$

<u>Pf</u> Since $S_k(\Gamma)$ is generated by P_m's it is enough to verify it for $f = P_m$ & $g = P_\ell$.

<u>Cor 6.14</u> says $\quad m^{k-1} T_n P_m = n^{k-1} T_m P_n \quad$ ①.

<u>Prop 6.15</u> says $\quad \ell^{k-1}\langle T_n f, P_\ell\rangle = n^{k-1}\langle T_\ell f, P_n\rangle \quad$ ②

Apply ② with $f = P_m$ to get

$$\ell^{k-1}\langle T_n P_m, P_\ell\rangle \overset{②}{=} n^{k-1}\langle T_\ell P_m, P_n\rangle \quad / \times m^{k-1}$$

$$\Longrightarrow \ell^{k-1} m^{k-1}\langle T_n P_m, P_\ell\rangle = n^{k-1} m^{k-1}\langle T_\ell P_m, P_n\rangle$$

$$\ell^{k-1} m^{k-1}\langle T_n P_m, P_\ell\rangle = n^{k-1} m^{k-1}\langle T_\ell P_m, P_n\rangle$$

$$\overset{①}{=} n^{k-1}\ell^{k-1}\langle T_m P_\ell, P_n\rangle$$

$$\overset{②}{=} \ell^{k-1} m^{k-1}\langle T_n P_\ell, P_m\rangle$$

$$\Longrightarrow \langle T_n P_m, P_\ell\rangle = \langle T_n P_\ell, P_m\rangle$$

Now $\langle T_n P_\ell , P_m \rangle = m$-th coef. of $T_n P_\ell$

This is real since coefs of P_ℓ are real

Hence $\langle T_n P_\ell , P_m \rangle = \overline{\langle T_n P_\ell , P_m \rangle}$

$$= \langle P_m , T_n P_\ell \rangle$$

since $\overline{\langle f, g \rangle} = \langle g, f \rangle$

and we get $\langle T_n P_m , P_\ell \rangle = \langle P_m , T_n P_\ell \rangle$

and T_n's are self-adjoint wrt $\langle \, , \rangle$

$\boxtimes$

__Thm 6-17:__ The space $S_k(\Gamma)$ of cusp forms has an orthonormal basis consisting of eigenfunctions of all Hecke operators T_n.

$\boxtimes$

Recall cor 6-13 says E_k is an e-func $\forall T_n$. Using this we have

__Thm 6-18__ Let $f \in S_k$, an eigenform $\forall T_n$. Then $\langle f, E_k \rangle =$

__Proof__ wlog assume f is normalized, $f(z) = \sum c(n) q^n$

Assume on the contrary $\langle E_k , f \rangle \neq 0$

$T_n E_k = \sigma_{k-1}(n) E_k \qquad T_n f = c_n f \qquad E_k = 1 - \frac{2k}{B_k} \sum \sigma_{k-1}(n) q^n$

then $\sigma_{k-1}(n) \langle E_k , f \rangle = \langle T_n E_k , f \rangle$

$$= \langle E_k , T_n f \rangle = \overline{c_n} \langle E_k , f \rangle$$

If $\langle E_k, f\rangle \neq 0$

then this gives $\sigma_{k-1}(n) = \overline{c_n}$

But since T_n's Hermitian, their eigenvalues are real. Hence $\overline{c_n} = c_n$

And therefore $\sigma_{k-1}(n) = c_n \quad \forall n$.

But then $E_k - \frac{2k}{B_k} f =$ constant and a modular form of wt k ↯

Another coor. of 6.17 is

Coor 6.19 The eigenvalues of T_n are algebraic numbers.

Proof. First note that the eigenvalues of T_n are real

If $T_n f = \lambda_n f$, then $\langle T_n f, f\rangle = \langle \lambda_n f, f\rangle$
$$= \lambda_n \langle f, f\rangle$$

On the other hand $\langle T_n f, f\rangle = \langle f, T_n f\rangle$
$$= \overline{\lambda_n}\langle f, f\rangle$$

Hence $\lambda_n = \overline{\lambda_n}$ and $\lambda_n \in \mathbb{R}$.

We have seen that $E_4^{\alpha} E_6^{\beta}$ with $4\alpha + 6\beta = k$ form a basis for $M_k(\Gamma)$

Recall $E_k(z) = 1 - \frac{2k}{B_k}\sum \sigma_{k-1}(n) q^n$ has rational coefs.

Hence $S_k(\Gamma)$ has a basis say $g_1, \dots, g_d$ such that each g_i has rational coefs.

Since $T_n(g_i)$ has again rational coefs

This means that we can represent T_n by a rational matrix in the basis $\{g_1 \dots g_d\}$

Therefore its characteristic poly. has rational coefs and the eigenvalues of T_n (the roots of this poly w/ rational coefs) are algebraic numbers. $\boxtimes$

Remark. One can in fact show that $S_k(\Gamma)$ has a basis with integer coefs. and consequently the matrix of T_n has $\mathbb{Z}$-coe[fs] and the eigenvalues of T_n are in fact algebraic integers.

For an example of this in action see série 4, exercise 4.